HYPERSONIC FLOWS OF RAREFIED GAS

(O GIPERZVUKOVYKH TECHENIIAKH RAZREZHENNOGO GAZA)

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M. N. KOGAN
(Moscow)

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We will examine here hypersonic flows close to the free-molecular type, which may be calculated only from the primary intermolecular collisions.

For a flow to be free-molecular, or close to it, it is not enough for the Knudsen number to be simply much greater than unity. The limit of free-molecular flows depends both on the shape of the body around which the flow occurs and on the velocity of the flow, the character of the interaction of the molecules with the surface of the body and among themselves. While, under certain conditions, the flow of a gas at an unlimited increase in velocity is not free-molecular, no matter how low its density, under others, at a rather high velocity, the flow will be free-molecular or close to it, no matter how great the density.

The phenomenon of "extraction" and "transformation" of the molecules has already been established; this allows us to calculate flows on the basis of primary collisions alone, even when one of the characteristic lengths of the molecular free paths is much less than the dimensions of the body.

It has been shown that under specific conditions and at high Reynolds' numbers, a thin molecular boundary layer forms around a plate exposed to flow at zero or small angle of incidence. In this article we establish that the drag of the plate at a small angle of incidence and also that of a thin cone, calculated on the basis of primary collisions, is greater than in the free-molecular flow.

We have also derived criteria for the existence of flows close to the free-molecular. For several characteristic cases we have given the form of the correction for drag and heat transfer produced by the primary collisions.

In Sections 1 and 2 we introduce the basic concepts and state the problem. Section 3 is devoted to the flow around a plate, placed perpendicular to the hypersonic flow. Section 4 treats an analogous problem for a plate parallel to the flow; here emerges the phenomenon of the molecular boundary layer. In Section 5 we examine the flow around an inclined plate and a cone.

1. Most frequently, a flow is considered free-molecular, if the length of the mean free path of the gas molecules at a given altitude (and at a given temperature and density) is much greater than the characteristic dimensions of the body [1]. The relationship between the length of the mean free path at a given altitude λ_0 and the characteristic dimensions L, designated by the Knudsen number, is for free-molecular flows assumed [1] equal to or greater than 10; i.e. $K = \lambda_0/L \gg 10$. The mean free path λ_0 for a gas with a given temperature and density is determined by the expression

$$\lambda_0 = c\,(n\sigma c^\circ)^{-1} \qquad (1.1)$$

where c is the molecular velocity in coordinates fixed in the gas, c° the average relative molecular velocity, n the number of molecules per unit of volume, and σ the cross-section of a molecule. For a Maxwell distribution, the velocities c and c° are of the same order and approximately equal to the velocity of sound in the gas a. We may thus write the relationship (1.1) in the form

$$\lambda_0 \sim (n\sigma)^{-1} \qquad (1.2)$$

The values c, c° and λ_0 are characteristics of the thermal agitation of the molecules in the gas. These values govern the processes of transger in the gas; in particular, for instance, the coefficient of viscosity μ is linked to these values by the relationship

$$\mu \sim mna\lambda_0 \sim \rho a\lambda_0 \qquad (1.3)$$

Here we replace the average velocity of molecular agitation by the velocity of sound, which is of the same order; ρ is the density of the gas.

However, if we examine the movement of the gas in terms of a fixed system of reference (for example, the flow in a nozzle), we observe, not only the values c and λ_0, which characterize the molecular movement with respect to the gas, but also new values characterizing the movement of gas molecules with respect to the fixed reference system. As examples of such characteristic values we may take the velocity of the gas with respect to the reference system V, or we may take the free path λ, traversed by the molecules between collisions in this system of coordinates. It is clear that at low velocities ($V \lesssim a$) the characteristic length of the mean free path will be λ_0, as before. At hypersonic velocities, however, it is easy to see that

$$\lambda \sim M\lambda_0 \qquad (M = V / a \gg 1) \tag{1.4}$$

or, substituting λ_0 with the aid of (1.3), we obtain

$$\frac{\lambda}{L} \sim \frac{M^2}{R} \sim MK \qquad \left(R = \frac{VL\rho}{\mu}, \quad K = \frac{\lambda_0}{L} \sim \frac{M}{R}\right) \tag{1.5}$$

where L is the characteristic dimension of the flow, R the Reynolds number and K the Knudsen number.

In hypersonic flow, therefore, the molecules travel between collisions a longitudinal distance which is M times greater than the transverse distance. In other words, we may characterize this anisotropy of hypersonic flow in the following manner: hypersonic flow is more rarefied in the direction of the flow than perpendicular to it.

Because of this anisotropy, the equations of a continuous medium (Navier-Stokes equations) at hypersonic velocities are true only for small longitudinal gradients, at the same time allowing significant transverse gradients (for instance, in the boundary layer).

Accroding to (1.5), the length of the mean free path increases proportionally to M^2; this creates difficulties in the construction of hypersonic wind tunnels.

2. The flow around a body in near free molecular flow depends on a number of characteristic mean free paths: the mean free path of the oncoming molecules in the field of molecules reflected from the body, λ_{12}; the path λ_{21} of re-emitted molecules in the oncoming stream field; the path of re-emitted molecules in the field of re-emitted molecules and also $\lambda_{11} = \lambda$ and λ_0. These paths, in turn, depend on the interaction of the molecules.

Generally speaking, the effective collision cross-section of σ molecules decreases as the relative velocity of the molecules increases. Some idea of the character of the relationship between σ and the relative velocity may be obtained from Formula (1.3).

From (1.3) it is obvious that $\mu \sim \rho \sqrt{(T)}\lambda_0 \sim \sqrt{(T)}/\sigma$, that is

$$\sigma \sim AT^{1/2}\mu^{-1} \tag{2.1}$$

Here and in the future, A designates a certain constant. If we assume Sutherland's law of the variation of viscosity with temperature, we obtain

$$\sigma = A(1 + S/T) \tag{2.2}$$

The Sutherland S constant for such gases as nitrogen, oxygen, helium and hydrogen lies in the range 80-140.

Let us designate by σ_∞ the collision cross-section in an equilibrium gas at a flow temperature T_∞, corresponding to a relative molecular velocity of the order of a_∞.

If σ is the collision cross-section at a relative velocity V

$$\frac{\sigma}{\sigma_\infty} = \frac{T_\infty}{T}\frac{T+S}{T_\infty+S} \quad \textbf{when} \quad \frac{T}{T_\infty} \sim \frac{V^2}{a_\infty{}^2} \sim M^2 \tag{2.3}$$

Thus, the cross-section in the earth's atmosphere will vary only by one-third when the relative velocity changes from 5 x 10^4 cm/sec (i.e. T = 300°C) to 10^6 cm/sec and above; in other words, the collision cross-section may be considered almost constant.

More substantial variations in σ may be expected in wind tunnels. Thus, in a helium tunnel T_∞ may be 5-10°K, while $T \geqslant 300$°K. Under these circumstances, the cross-section decreases by one order or more.

In order to illustrate the influence of the variation of the collision profile as a function of the relative velocity, we will consider below two characteristic cases; σ = const and σ inversely proportional to the relative velocity.

We will assume that the repulsion of molecules from the plate is diffuse with a Maxwell distribution of velocities. The average velocity of the re-emitted molecules is determined from the temperature of the wall T_w and the accommodation coefficient α.

3. Let us examine the hypersonic flow of a rarefied gas past a plate of characteristic dimension L at a velocity V_∞ perpendicular to the plane of the plate. The Mach number $M \gg 1$ and the mean free path of the flow is equal to λ_0.

The thermal agitation velocities of the molecules of the inflow in that case are negligible; we may disregard them and consider that a cluster of molecules in uniform parallel motion strikes the plate at a velocity V_∞. The mean free path $\lambda_0 \sim (n_\infty\sigma)^{-1}$ involved here characterizes only the density of the incoming molecules.

Here, the determining dimensionless flow parameter is the ratio $M_2 = V_\infty/V_2$ of the velocity of the inflow V_∞ to the average velocity V_2

of the re-emitted molecules, since it is this ratio which determines the density of the re-emitted molecules

$$n_2 \sim n_\infty V_\infty / V_2 \sim n_\infty M_2 \tag{3.1}$$

and, consequently, also the mean free paths λ_{12}, λ_{21} and λ_{22}.

Let us now examine the cases $M_2 \gg 1$ and $M_2 \sim 1$.

3.1.1. Let $M_2 \gg 1$ and $\sigma = \sigma_\infty = \text{const}$. This case may be realized, for example, with diffuse re-emission with an accommodation coefficient $\alpha \sim 1$ and $T_w \sim T_\infty$; under these conditions $M_2 \sim M$, since $V_2 \sim a_\infty$.

The characteristic mean free paths here are equal to

$$\lambda_{12} \sim V_\infty (n_2 \sigma_\infty V_\infty)^{-1} = \lambda_0 M_2^{-1}, \qquad \lambda_{21} \sim V_2 (n_\infty \sigma_\infty V_\infty)^{-1} \sim \lambda_0 M_2^{-1}$$
$$\lambda_{22} \sim V_1 (n_2 \sigma_\infty V_2)^{-1} \sim \lambda_0 M_2^{-1} \tag{3.2}$$

since the relative velocities $V_{12} \sim V_{21} \sim V_\infty$.

In order for the flow to be free-molecular, or close to it, the molecules must sustain few collisions within distances of the order of L from the plate; i.e. it is necessary that [2,3]

$$\lambda_{12} / L = \lambda_{21} / L = \lambda_{22} / L = K / M_2 \gg 1 \tag{3.3}$$

If this condition is satisfied, then to obtain the first correction for free-molecular flow we need but consider the first collisions of the molecules.

Let us designate by $N_0 = n_\infty V_\infty L^2$, $P_0 \sim N_0 m V_\infty$, and $Q_0 \sim N_0 m V_\infty^2$ respectively the number of molecules striking the plate per unit time in free-molecular flow, the momentum they carry with them, and their energy. Let us further designate by N_-, P_- and Q_- respectively the decrease, due to collisions, in the number of particles, in the momentum they carry, and in their energy. Each particle in the flow sustains $n_2 \sigma L$ collisions. Therefore

$$N_- \sim N_0 n_2 \sigma L \sim N_0 M_2 K^{-1}, \qquad P_- \sim N_- m V_\infty, \qquad Q_- \sim N_- m V_\infty^2 \tag{3.4}$$

It is easy to verify that the increase in these quantities (N_+, P_+, Q) attributable to the additional molecules which strike the plate as a result of the collisions is of the same order.

The change in the energy and momentum delivered to the plate as a result of collisions of re-emitted molecules with each other is negligible, because of their small velocity and of the small probability of their return to the plate after collision. Thus

$$
\begin{aligned}
C_x &= 2\frac{P_0 - P_- + P_+}{\rho_\infty V_\infty{}^2 L^2} \sim C_{x0} + \frac{AM_2}{K} + O(M^2K^{-1}, K^{-1}) \\
q &= 2\frac{Q_0 - Q_- + Q_+}{\rho_\infty V_\infty{}^3 L^2} \sim q_0 + \frac{AM_2}{K} + O(M^2K^{-2}, K^{-1})
\end{aligned}
\tag{3.5}
$$

where C_x is the drag coefficient and q the heat flow, while the index 0 refers to free-molecular flow and A is a constant.

From the geometry of the flow it is obvious that the majority of the collisions occur near the body, and the greater part of the molecules involved in these strike the plate, carrying the same momentum and energy as if there had been no collision. Thus, for a plate perpendicular to the flow, the constant A must be small. Calculations [4] indicate that A is negative.

3.1.2. Once more, let $M_2 \gg 1$, σ is inversely proportional to the relative velocity of the molecules, and $T_w \sim T_\infty$. Under these circumstances

$$\sigma_2 = \sigma_\infty M_2{}^{-1}, \qquad \lambda_{12} \sim \lambda_0, \qquad \lambda_{21} \sim \lambda_0, \qquad \lambda_{22} \sim \lambda_0 M_2{}^{-1}$$

If $\lambda_{22} \gg L$; i.e. if $KM_2^{-1} \gg 1$, then in the vocinity of the body few collisions take place between the incoming molecules and the re-emitted molecules or between the re-emitted molecules themselves, and we have, as in 3.1.1

$$C_x = C_{x0} + AK^{-1} + O(M_2{}^{-2}K^{-1}) \qquad (K \gg M_2) \tag{3.6}$$

An analogous expression is obtained for the heat flow.

In the other limiting case; that is, when $\lambda_{22} \ll L$ or $1 \ll K \ll M_2$, the re-emitted molecules sustain a large number of collisions among themselves at a small distance from the plate, but practically no collisions with the incoming molecules. At a distance of several mean free paths λ_{22} from the plate, the flow of re-emitted molecules may be regarded in the same way as an outflow of a gas into vacuum, from which the incoming stream "extracts" molecules, "transforming" them, as it were, into molecules of a different type with a mean free path of the order of $\lambda_0 \sim \lambda_{22}M$ (since during a collision the velocity, and consequently also the mean free path of the re-emitted molecules increases an average of M times), and at the same time creating small disturbances in this gas. Thus, the flow of a continuous medium of re-emitted molecules interacts, as it were, on inflow of molecules of a different type, characterized by a large mean free path in the field of re-emitted molecules. A basic difficulty in the analysis of this interaction is the study of the character of the outflow of the re-emitted molecules. But this problem has an interest all of its own and will not be considered in the present article.

3.2.1. Now let $M_2 \sim 1$ and σ = const. This case may arise when the accommodation coefficient $\alpha \ll 1$ (in practice $\leqslant 0.5$), when $T_w/T_\infty \sim 1$, or when the wall is heated to a temperature of the same order as the stagnation temperature. This last case is often observed in wind tunnel studies.

It is clear that under these conditions $n_2 \sim n_\infty$, and

$$\lambda_{12} \sim V_\infty (n_\infty \sigma V_\infty)^{-1} \sim \lambda_0, \qquad \lambda_{21} = V_\infty (n_\infty \sigma V_\infty)^{-1} \sim \lambda_0$$
$$\lambda_{22} \sim V_\infty (n_\infty \sigma V_\infty)^{-1} \sim \lambda_0 \qquad (3.7)$$

The criterion of similarity will be the number K. When $K \gg 1$, we have

$$C_x = C_{x0} + AK^{-1} + O(K^{-2})$$

3.2.2. If $M_2 \sim 1$ and σ varies in inverse proportion to the relative velocity (see 3.1.2), $\sigma_2 \sim \sigma_\infty M^{-1}$ and, consequently

$$\lambda_{21} \sim \lambda_{12} \sim \lambda_{22} \sim M\lambda_0 \qquad (3.8)$$

i.e. the criterion of similarity will be MK. When $MK \gg 1$

$$C_x = C_{x0} + A(KM_2)^{-1} + O(K^{-1}M^{-2}, K^{-2}M^{-2})$$

We note that at large enough Mach numbers this flow may be realized at $K \ll 1$; i.e. in a dense medium. When $MK \gg 1$, the character of the flow around the plate will be close to free-molecular, even when the inflow is regarded as continuous.

Here, as in 3.1.2, the re-emitted molecules and the molecules which have collided with re-emitted molecules will be like molecules of a new type, possessing a mean free path M times greater than the molecules in the inflow. The re-emitted molecules, on colliding with the incoming molecules "snatch out" from the incoming stream molecules which then become of a different type. The departure of these molecules from the incoming stream creates disturbances which propagate at sonic velocity a_∞. However, during the time required for the gas to travel a distance of the order of L, these disturbances cover a path $L/M \ll \lambda_0$. Therefore we may disregard changes in the incoming stream due to gas-dynamic effects. This is particularly clear if we represent the inflow in the form of a rather dense cluster of molecules in parallel motion.

It is interesting that, depending on the conditions, the criterion of similarity changes from K/M in case 3.1.1 to MK in case 3.2.2, or by a factor M^2.

If, in the first case, at any density of inflow we increase the velocity indefinitely, the character of the flow will tend to continuum; in the second case the flow will tend toward the free-molecular.

In accordance with these criteria, a shift is also observed in the limit of free-molecular flows. It is obvious that in case 3.1.1 the free-molecular flow is realized at the same Mach number but at a much higher altitude than in case 3.2.2.

4. Let us examine the other limiting case of a plate exposed to flow at a zero angle of incidence. In the free-molecular flow involved here, the molecules strike the plate only at thermal velocities. The number of molecules striking the plate is $N_0 \sim L^2 n_\infty a_\infty$.

4.1.1. Let $M \sim M_2 >> 1$ and σ = const. Under these conditions, $n_2 \sim n_\infty$, $V_2 \sim a_\infty$ and

$$\lambda_{12} \sim \lambda_0, \qquad \lambda_{21} \sim \lambda_0 M^{-1}, \qquad \lambda_{22} \sim \lambda_0 \tag{4.1}$$

If $\lambda_{21} \sim \lambda_0 M^{-1} >> L$; i.e. $K >> M$, every molecule of the mean stream sustains $n_\infty \sigma L$ collisions near the body. In all, near the body, a total of $N_+ \sim n_\infty^2 V_\infty \sigma L^3$ collisions take place, of which $N_- \sim N_0 n_\infty \sigma L \sim n_\infty^2 a_\infty \sigma L^3$ collisions are sustained by molecules striking the body in a free-molecular flow. Hence, under these conditions, $N_- \sim N_+ M^{-1} \ll N_+$. Thus, the collisions cause additional momentum and more energy to be delivered to the plate.

The drag in this case is

$$C_x = C_{x0} + AK^{-1} + O(M^{-1}K^{-1}), \qquad C_{x0} = O(M^{-1}) \tag{4.2}$$

where the first term is determined by the momentum delivered by the molecules in free-molecular flow, and quantity A is positive.

Thus the drag, and also the heat flow, are larger in the presence of collisions than in free-molecular flow.

Now let $\lambda_{21} \ll L$ and $1 \ll K \ll M$. Here, all the re-emitted molecules sustain collisions with incoming molecules at distances of the order of $\lambda_0/M \ll L$ from the plate.

As a result of their collisions with incoming molecules, the re-emitted molecules attain an average velocity of the order of V_∞. Their mean free path increases M times, so that they sustain only one collision apiece within a boundary layer of width of order $\lambda_0 M^{-1}$.

Let us designate the number of molecules leaving the plate per unit time by N_w. Since the velocity of the re-emitted molecules $a_\infty \ll V_\infty$, approximately half the molecules participating in the collisions proceed on to the upper boundary of the layer, and the other half to the wall. Thus, we observe a return to the wall of what are almost N_w molecules. Outflow from the layer is confined to its ends. On the other hand, $N_0 - N_-$ molecules from the undisturbed flow enter the layer per unit time. For this reason, the density of the molecules in the layer increases until the outflow at the ends of the layer begins to balance the molecular inflow. Let that portion of the molecules which leaves the layer at its ends be equal to εN_w. We may then roughly say that the value of ε is proportional to the ratio between the area of the layer ends and that of the plate; i.e. $\varepsilon \sim KM^{-1}$. In that case, we have

$$N_0 - N_- \sim N_w \varepsilon \tag{4.3}$$

The density of the re-emitted molecules in the layer $n_2{}^* \sim N_w a^{-1} L^{-2}$. The mean free paths in the layer are correspondingly

$$\lambda_{12}{}^* \sim (n_2{}^* \sigma)^{-1} \sim aL^2 (N_w \sigma)^{-1}, \qquad \lambda_{21}{}^* \sim \lambda_0 M^{-1}$$
$$\lambda_{22}{}^* \sim (n_2{}^* \sigma)^{-1} \sim aL^2 (N_w \sigma)^{-1} \tag{4.4}$$

In calculating the path lengths $\lambda_{21}{}^*$ and the layer thickness which is of the same order we assume that the density of the incoming molecules is of the order of n_∞ throughout the whole layer. This condition is fulfilled when $\lambda_{12}{}^* \gg L$ or, according to (4.4), when

$$aL (N_w \sigma)^{-1} \gg 1 \tag{4.5}$$

When $\lambda_{12}{}^* \gg L$, it is obvious that $N_- \ll N_0$ and that, in conformity with (4.3)

$$N_w \sim N_0 \varepsilon^{-1} \sim N_0 M K^{-1}$$

Substituting this expression into (4.5), we obtain $K^2 \gg M$.

Thus, for the realization of the proposed flow scheme, it is necessary that $K^2 \gg M \gg K \gg 1$. It is clear that these conditions can be fulfilled only at very high Knudsen and Mach numbers.

Several other examples might be cited in addition to the above.

Let us observe a flow of particles of density n_∞, moving at a velocity V_∞ past a plate. Let this flow strike the plate at a small angle θ. Furthermore, let the velocity of the re-emitted molecules $V_2 \ll V_\infty$; i.e.

let $M_2 \ll 1$. Under these conditions, the thickness of the layer is equal to $KM_2^{-1}L$, $\lambda_{12}^* \sim \lambda_0 K (\theta M_2^2)^{-1}$, and the existence conditions for the layer are $K \ll M$ and $K^2 \gg \theta M_2^2$.

These conditions may also be found in re-emission arrangements of a different nature: molecular beam or similar type.

When the above conditions are fulfilled, then even at Knudsen numbers, much greater than unity, the flow differs substantially from the free molecular, and the drag and heat flow, as we will show below, may exceed the corresponding free-molecular values by several orders of magnitude.

In the above discussion we did not consider collisions outside the molecular layer. As noted above, a number N_w of molecules leave through the upper boundary of the layer with an average velocity V_∞ and a number $N_w\varepsilon$ of molecules through the ends of the layer with an average velocity a_∞. The density of these flows is $n_2 = N_w(V_\infty L^2)^{-1} \sim n_\infty K^{-1}$ and n_2^*, respectively.

Molecules which have left through the upper boundary sustain $n_\infty V_\infty n_2 \sigma L^3 \sim N_0 MK^{-2} \ll N_0$ collisions. Those which have emerged through the layer ends sustain $N_0 KM^{-1}$ collisions; consequently the role played by these collisions is less significant.

Thus, as a result of all these phenomena taking place in the layer, a total of $N_w \sim N_0 MK^{-1}$ molecules strikes the plate per unit time, whereas in free-molecular flow only N_0 molecules strike it. After the collision of N_w incoming molecules, carrying a momentum $N_w mV_\infty$ and an energy $1/2\, N_w mV_\infty$, with N_w re-emitted molecules, whose momentum and energy may be disregarded here, half of the momentum and energy is carried away by molecules departing through the upper boundary of the layer and half strikes the plate. Therefore

$$C_x \sim A_0 M^{-1} + {}^1/_2 A_1 K^{-1} \tag{4.6}$$

Thus, at high Reynolds' numbers $R = MK^{-1}$ in the interval $K \gg R \gg 1$, the drag and the heat flow, determined by a term of order K^{-1}, are R times greater than in free-molecular flow.

In actual practice, we may weaken condition (4.5) and require only $L \leqslant \lambda_{12}^*$; i.e. $K^2 \geqslant M$, since the resulting decrease in the inflow can be no more than ε times as great. However, these estimates are given without taking orders strictly into account.

In the intermediate area; i.e. when $K \sim M$, all the re-emitted molecules sustain collisions at distances of order L from the plate. Considerations analogous to the above yield

$$C_x \sim A_0 M^{-1} + A_1 K^{-1} \tag{4.7}$$

Thus, the calculation of hypersonic flow around a plate in a range of Knudsen numbers from $K \gg M$ to $K^2 \gg M \gg K$ (from $R \ll 1$ to $\sqrt{M} \gg R \gg 1$) may be carried out strictly on the basis of first collisions of molecules, in spite of the fact that one of the characteristic mean free path lengths λ_{21} is comparable, or even much less than the characteristic dimensions of the body. When even larger numbers R are involved, it is necessary to take into account secondary and other collisions as well.

4.1.2. Again let $M \sim M_2 \gg 1$ and σ be inversely proportional to the relative velocity of the molecules. Under these conditions, $\sigma_2 \sim \sigma_\infty M^{-1}$, $V_2 \sim a_\infty$, $n_2 \sim n_\infty$

$$\lambda_{12} \sim M\lambda_0, \qquad \lambda_{21} \sim \lambda_0, \qquad \lambda_{22} \sim \lambda_0 \tag{4.8}$$

Let us examine three cases: $\lambda_0 \gg L$, $\lambda_0 \sim L$ and $\lambda_0 \ll L$, when $MK \gg 1$.

If $\lambda_0 \gg L$, then in the vicinity of the plate $n_\infty{}^2 V_\infty \sigma_\infty M^{-1} L^3$ collisions between inflow and re-emitted molecules will occur, and

$$C_x \sim A_0 M^{-1} + A_1 (KM)^{-1} \tag{4.9}$$

The correction necessitated by the collisions between re-emitted molecules and by the loss in momentum sustained by the incoming molecules colliding with re-emitted molecules is of order $(KM^2)^{-1}$. Once again the drag is larger than in free-molecular flow. When $K \geqslant M$, we have to consider the momentum carried off by re-emitted molecules to be of order M^{-2}.

When $\lambda_0 \sim L$, all the re-emitted molecules sustain collisions both with each other and with incoming molecules at distances of order L. Collisions between the re-emitted molecules themselves do not alter the density or the velocity of these molecules. However, a collision of a re-emitted molecule with an incoming molecule "transforms", as it were, the re-emitted molecule into a molecule of a different type, with a mean free path equal to $\lambda_0 M$. By analogy to 4.1.1, we have

$$C_x = A_0 M^{-1} + A_1 (KM)^{-1} \sim A_0 M^{-1} + A_2 M^{-1} \tag{4.10}$$

since, in the present instance, $K \sim 1$, and $MK \gg 1$.

In the limiting case $\lambda_0 \ll L$, all re-emitted molecules collide with incoming molecules in a narrow layer, of width λ_0, in the vicinity of the plate. Through procedures similar to those used for the case of the molecular boundary layer in 4.1.3, we obtain $N_w \sim N_0 K^{-1}$, $n_2{}^* \sim n_\infty K^{-1}$ and

$$\lambda_{12}{}^* \sim \lambda_0 KM, \qquad \lambda_{21}{}^* \sim \lambda_0, \qquad \lambda_{22}{}^* \sim \lambda_0 K \tag{4.11}$$

From condition $\lambda_{12}{}^* \gg L$ emerges the requirement $MK^2 \gg 1$, where $K \ll 1$. For the drag, we obtain

$$C_x \sim A_0 M^{-1} + A_1 (KM)^{-1} \tag{4.12}$$

Here, once more, the drag is much larger than in free-molecular flow. However, the calculation of this flow cannot be carried out on the basis of first collisions alone, since $\lambda_{22}{}^* \ll \lambda_0$ and thus in the layer there occur many collisions among the re-emitted molecules.

In this case, as in 3.2.2, the flow is close to free-molecular in character during movement in a dense medium ($K \ll 1$).

The process of "transformation" of the re-emitted molecules into molecules with a larger mean free path is similar to that in 4.1.1. The situation in 4.1.1, however, was more favorable for calculation, since the re-emitted molecules in the layer did not manage to collide with each other. In that calculation, therefore, it was sufficient to consider merely the decrease in the flow of re-emitted molecules due to the transformations resulting from collisions with incoming molecules. But in 4.1.2 collisions among re-emitted molecules occur, and alter the distribution function in a complicated way.

In the above cases, once again, the drag and heat transfer curves have a maximum with respect to Knudsen number.

4.2.1. Let us examine the case of high velocity re-emitted molecules $M_2 \sim 1$; i.e. $V_2 \sim V_\infty$ and σ = const. In this instance, $n_2 = n_\infty M^{-1}$, and

$$\lambda_{12} = M\lambda_0, \qquad \lambda_{21} \sim \lambda_0; \qquad \lambda_{22} = M\lambda_0 \tag{4.13}$$

This case is similar to 4.1.1 in that the shortest mean free path is λ_{21}. However, in the present instance all the molecules have the same order of velocity V_∞ and therefore undergo no transformation.

The flow here is close to free-molecular only when all mean free paths are much larger than L. Consequently, K must $\gg 1$ and

$$C_x \sim C_{x0} + A_1 (KM^2)^{-1} \tag{4.14}$$

4.2.2. When $M_2 \sim 1$ and σ is inversely proportional to the velocity

$$\lambda_{12} \sim M^2\lambda_0, \qquad \lambda_{21} \sim M\lambda_0, \qquad \lambda_{22} \sim M^2\lambda_0 \tag{4.15}$$

As in 4.2.1, the existence condition for flow close to free-molecular is $KM \gg 1$ and

$$C_x = C_{x0} + A_1 (KM^2)^{-1} \tag{4.16}$$

The flow is close to free-molecular, and the necessary correction may be found by considering only first collisions, where $K \ll 1$, if $KM \gg 1$. In both 4.2.1 and 4.2.2, the drag is greater than in free-molecular flow.

It should be observed that, when $M_2 \sim 1$, the probability that a molecule will strike the wall after it has sustained a collision is smaller than when $M_2 \gg 1$. For this reason, the numerical value of the coefficients A_1 is correspondingly less.

5. For a plate exposed to molecules at arbitrary angles of incidence ϑ, just as was true in the theory of hypersonic flows of a continuous medium, it is necessary to distinguish two cases: $M \sin \vartheta \gg 1$ and $M \sin \vartheta \ll 1$.

In the first case ($M \sin \vartheta \gg 1$), the character of the flow is basically similar to that at $\vartheta = \pi/2$, since the presence of thermal velocities is not essential for the calculation of the number of particles and the momentum and energy they deliver. The number of molecules striking the wall is $N_0 = V_\infty n_\infty L^2 \sin \vartheta$.

In Sections 3 and 4 we demonstrated that at $\vartheta = \pi/2$ the drag (heat transfer) was smaller, and at $\vartheta = 0$ correspondingly greater than in free-molecular flow. In the present case, $n_2 \sim M_2 n_\infty \sin \vartheta$, and

$$\lambda_{12} \sim \lambda_0 (M_2 \sin \vartheta)^{-1}, \qquad \lambda_{21} \sim \lambda_0 M_2^{-1}, \qquad \lambda_{22} \sim \lambda_0 (M_2 \sin \vartheta)^{-1} \tag{5.1}$$

When $\lambda_{21} \gg L$, or $K \gg M_2$, it is easily seen that

$$N \sim n_\infty^2 V_\infty M_2 \sigma_\infty L^3 \sin^2\vartheta, \qquad N_- \sim n_\infty^2 V_\infty M_2 \sigma_\infty L^3 \sin\vartheta$$

$$C_x \sim A_0 \sin\vartheta - A_1 M_2 K^{-1} \sin^2\vartheta + A_2 M_2 K^{-1} \sin\vartheta \tag{5.2}$$

When $\vartheta \to 1/2\,\pi$, the second negative term is greater than the third, and the drag is less than in free- molecular flow. When $\vartheta = 0$, the situation is reversed.

When ϑ is small and $K \leqslant M_2$, all re-emitted molecules sustain collisions at distances $LKM_2^{-1} \ll 1$ from the plate. The drag is described by the same formula (5.2). At the limit of $KM_2^{-1} \ll 1$, we arrive at a boundary layer of the type described in Section 4.

When $M_2\vartheta \ll 1$, the flow is similar to the flow when $\vartheta = 0$.

Flow around a cone may be considered in the same way. As with the plate, it is easy to show that the drag of the cone is

$$C_x \sim C_{x0} - A_1MK^{-1}\vartheta + A_2MK^{-1}$$

where ϑ is the semi-angle of the span of the cone. Consequently, at a rather small span angle, the drag of the cone, considered on the basis of first collisions, is larger than in free-molecular flow.

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BIBLIOGRAPHY

1. Tsien, H.S., *Aerodinamika razrezhennykh gazov (Aerodynamics of Rarefied Gases)*. Sb. per. Gazovaia dinamika, IIL, 1950.

2. Kogan, M.N., *O granitse svobodnomolekuliarnykh techenii (The Limit of Free-molecular Flow)*. Tr. TsAGI, No. 764, 1959.

3. Hayes, W.D. and Probstain, R.F., *Hypersonic Flow Theory*. Academic Press, N.Y., 1959.

4. Perepukhov, V.A., O soprotivlenii ploskoi plastinki v potoke sil'no razrezhennogo gaza (The resistance of a plane lamina in the flow of a highly rarefied gas). *Zhurn. vychislit. matem. i matem. fiz.* Vol. 1, No. 4, 1961.

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